

3.5 Dimension tangent spaces & residual gerbes.

§3.5.1 Dimension

Def'n 3.5.1

- 1) X scheme $\dim X :=$ Krull dimension of its underlying top. space
 $\dim_x X = \min \{ \dim U \mid U \subset X \text{ open, } x \in U \}$.
- 2) X Noetherian alg. space $x \in |X|$.
 $\dim_x X := \dim_x U \in \mathbb{Z} \cup \{+\infty\}$,
 where $U \rightarrow X$ is an étale pres. and $u \in U$ maps to x .
- 3) $\mathcal{X}$ Noeth. alg. stack w/ smooth rep. $U \rightarrow \mathcal{X}$ and corresponding sm. groupoid $s, t: R \rightarrow U$, $u \in U$ maps to $x \in |\mathcal{X}|$
 $\dim_x \mathcal{X} := \dim_u U - \dim_{e(u)} R_u \in \mathbb{Z} \cup \{+\infty\}$ where R_u is the fiber of $s: R \rightarrow U$ and $e: U \rightarrow R$ identity.
 $\dim \mathcal{X} := \sup_{x \in |\mathcal{X}|} \dim_x \mathcal{X} \in \mathbb{Z} \cup \{+\infty\}$.

Lem: 1) $\dim_x \mathcal{X}$ of a Noeth. alg. space is indep. on $U \rightarrow \mathcal{X}$.

2) $X \rightarrow Y$ sm. morphism of Noeth. alg. space

$$x \in |X| \mapsto y \in |Y| \text{ then}$$

$$\dim_x X = \dim_y Y + \dim_x X_y$$

3) $X, X' / k$ Noeth. alg. space w/ k -pts x, x' resp. then

$$\dim_{(x, x')} X \times_k X' = \dim_x X + \dim_{x'} X'$$

3) $\mathcal{X} / k$ Noeth. alg. space $k \rightarrow L$ field extension.

$$\mathcal{X}_L := \mathcal{X} \times_k L \quad x' \in |\mathcal{X}_L| \mapsto x \in |\mathcal{X}| \text{ then}$$

$$\dim_{x'} \mathcal{X}_L = \dim_x \mathcal{X}.$$

Prop 3.5.2: $\mathcal{X}$ Noeth. alg. stack $\dim_x \mathcal{X}$ is indep on

$U \rightarrow \mathcal{X}$ and the preimage u of x .

Pf: $U \rightarrow \mathcal{X}$
 $u \mapsto x$
 $R_u \rightarrow R \xrightarrow{s} U \quad R_u = U \times_{\mathcal{X}} \text{Spec } k(u)$
 $\downarrow \quad \downarrow \quad \downarrow$
 $\text{Spec } k(u) \rightarrow U \rightarrow \mathcal{X} \quad \text{is a sm. alg. space / } k(u)$

$$U' \rightarrow \mathcal{X} \quad u' \mapsto x \quad u' := U \times_{\mathcal{X}} U'$$

$$U''_u \rightarrow U'' \rightarrow U' \quad U''_u = U'' \times_U \text{Spec } k(u) = U' \times_{\mathcal{X}} U \times_U \text{Spec } k(u)$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\text{Spec } k(u) \rightarrow U \rightarrow \mathcal{X} = U' \times_{\mathcal{X}} k(u) \cong R'_u \quad u' \mapsto e'(u')$$

$$\text{Lem. 1) } \Rightarrow \dim_{u''} U'' = \dim_u U + \dim_{u''} U'' = \dim_u U + \dim_{e'(u')} R'_u$$

$\exists L/k$ s.t. $\text{Spec } L \rightarrow U'$ mapping to u, u' resp.

$$R''_{u''} := U'' \times_{\mathcal{X}} \text{Spec } L.$$

$$\text{WTS: } \dim_u U - \dim_{e(u)} R_u = \dim_{u''} U'' - \dim_{e'(u')} R'_u$$

$$\text{Symmetry } \Leftrightarrow \dim_u U - \dim_{e(u)} R_u = \dim_{u''} U'' - \dim_{e'(u')} R''_{u''}$$

where $e''(u'') \in |R''_{u''}|$ is the image of $\text{Spec } L \rightarrow R''_{u''}$

$$\stackrel{\text{Q.E.D.}}{\Leftrightarrow} \dim_{e''(u'')} R''_{u''} = \dim_{e(u)} R_u + \dim_{e'(u')} R'_u.$$

$$\begin{array}{ccc} R''_{u''} & \xrightarrow{\quad} & R'_u \times_{k(u)} \text{Spec } L \\ \downarrow & \searrow & \downarrow \\ U'' & \xrightarrow{\quad} & U' \\ \downarrow & \searrow & \downarrow \\ U & \xrightarrow{\quad} & \mathcal{X} \end{array} \quad \begin{array}{l} \text{Lem 2)} \\ \dim R''_{u''} = \dim_{e(u)} R_u \times_{k(u)} \text{Spec } L \\ + \dim_{e'(u')} R'_u \times_{k(u)} \text{Spec } L \\ \text{Lem 3)} \\ = \dim_{e(u)} R_u + \dim_{e'(u')} R'_u. \\ \text{as desired.} \end{array}$$

Eg: U/k scheme of pure dim $G \subset U$ affine alg. gp.

$$\dim [U/G] = \dim U - \dim G$$

$$\text{in particular } \dim BG = -\dim G.$$

§3.5.2 Tangent space $k[x] := k[x]/(x^2) =$ ring of dual #'s

X/k scheme. $x \in X(k)$

$$T_{X,x} := \left\{ \begin{array}{ccc} \text{Spec } k & \xrightarrow{x} & \\ \downarrow & \searrow & \\ \text{Spec } k[x] & \xrightarrow{\sim} & X \end{array} \right\} \quad k\text{-vector space.}$$

Def'n 3.5.5: $\mathcal{X}$ alg. stack $x: \text{Spec } k \rightarrow \mathcal{X}$

The Zariski tangent space $T_{\mathcal{X},x}$ of $\mathcal{X}$ at x is the set

$$T_{\mathcal{X},x} = \left\{ \begin{array}{ccc} \text{Spec } k & \xrightarrow{x} & \\ \downarrow \alpha & \searrow & \\ \text{Spec } k[x] & \xrightarrow{\sim} & \mathcal{X} \end{array} \right\} / \sim$$

$$= \{ (\tau, \alpha) \mid \tau: \text{Spec } k[x] \rightarrow \mathcal{X}, \alpha: x \rightsquigarrow \tau|_{\text{Spec } k} \}$$

where $(\tau, \alpha) \sim (\tau', \alpha')$ iff $\exists \beta: \tau \rightsquigarrow \tau'$ s.t. $\alpha' = \beta_{\text{Spec } k} \circ \alpha$.

Prop 3.5.6: $\mathcal{X}$ alg. stack w/ affine diag., $x \in \mathcal{X}(k)$

Then $T_{\mathcal{X},x}$ is naturally a k -vector space.

$$\text{Pf: } \forall c \in k \quad C \cdot (\tau, \alpha) := \left\{ \begin{array}{ccc} \text{Spec } k[x] & \xrightarrow{c \cdot \tau} & \mathcal{X} \\ \downarrow \alpha & & \\ \text{Spec } k[x] & \xrightarrow{\sim} & \mathcal{X} \end{array} \right\}$$

$$\text{Claim: } \begin{array}{ccc} \text{Spec } k & \hookrightarrow & \text{Spec } k[x] \\ \downarrow & & \downarrow \\ \text{Spec } k[x] & \hookrightarrow & \text{Spec } k[x] \times_k k[x] \end{array} \quad \begin{array}{l} \text{is a pushout for} \\ \text{alg. stacks w/} \\ \text{affine diagonal.} \end{array}$$

Granting the claim, define $(\tau, \alpha) + (\tau', \alpha')$ by

$$\text{Spec } k[x] \rightarrow \text{Spec } k[x] \times_k k[x] \rightarrow \mathcal{X} \quad \text{It is not hard to}$$

$$\tau \leftarrow \begin{pmatrix} \tau, 0 \\ 0, \tau' \end{pmatrix} \quad \text{verify axioms for v.s.}$$

Proof of claim: $(U, u) \rightarrow (X, x)$ sm. morphism from affine schm U .

$\mathcal{X}$ has affine diagonal $\Rightarrow U \rightarrow \mathcal{X}$ affine.

$$\text{Spec } A := \text{Spec } k \times_{\mathcal{X}} U \quad \text{Spec } A_i = \text{Spec } k[x_i] \times_{\mathcal{X}} U.$$

$$\text{Clearly } \text{Spec } A \rightarrow \text{Spec } A_1$$

$$\downarrow \quad \downarrow$$

$$\text{Spec } A_2 \rightarrow \text{Spec } (A_1 \times_k A_2)$$

$$\text{Spec } k[x_1, x_2] \dashrightarrow \mathcal{X} \quad \text{red arrows}$$

$A_1 \times_k A_2$ is free/ $k[x_1, x_2]$ $\Rightarrow$ faithfully flat $\Rightarrow$

Same argument works for $U \times_{\mathcal{X}} U$

$$\text{Spec } (A_1 \times_k A_2) \rightarrow U \text{ descends to } \text{Spec } k[x_1, x_2] \dashrightarrow \mathcal{X} \quad \square$$

Eg: $g \geq 2, [C] \in \mathcal{M}_g(k)$ Rmk: Prop true w.o. assuming affine diagonal.

$$T_{\mathcal{M}_g, [C]} = H^1(C, T_C) \quad \text{so}$$

$$\dim T_{\mathcal{M}_g, [C]} = h^1(C, T_C) = -\chi(T_C) = 3g-3.$$

§3.5.3 Residual gerbes

X scheme $x \in X$: $\exists k(x) \xrightarrow{x} X$ Eg: X scheme $x \in X$ pt

Def'n: $\mathcal{X}$ alg. stack $x \in |\mathcal{X}|$. We say that $x \hookrightarrow X$ is the residual gerbe.

the residual gerbe at x exists if

$\exists$ reduced Noeth. alg. stack $\mathcal{G}_x \hookrightarrow \mathcal{X}$ s.t.

$$| \mathcal{G}_x | \mapsto x \in |\mathcal{X}|.$$

In this case $\mathcal{G}_x$ is the residual gerbe at x .

Rmk: 1) $\mathcal{G}_x$ is unique

2) $\mathcal{G}_x$ is a gerbe over a field $k(x)$ (the residue field).

Def'n 3.5.13: $x \in |\mathcal{X}|$ in an alg. stack is of finite type if

$\exists$ a rep. $\text{Spec } k \rightarrow \mathcal{X}$ loc. of fin. type

Prop 3.5.16: $\mathcal{X}$ Noeth. alg. stack. $x \in \mathcal{X}$ a f.t. pt

then the residual gerbe $\mathcal{G}_x$ at x exists and

is a reg. alg. stack, and $\mathcal{G}_x \hookrightarrow \mathcal{X}$ is a loc. closed immersion.

If in addition $\mathcal{X}$ is of f.t./ k and $x \in \mathcal{X}(k)$ has

affine sm. stabiliser G_x then $\mathcal{G}_x = BG_x$.

Pf: Replacing $\mathcal{X}$ with $\overline{\mathcal{X}}$ wMA $\mathcal{X}$ reduced $x \in |\mathcal{X}|$ dense.

Let $\text{Spec } k \rightarrow \mathcal{X}$ be a f.t. rep. of x .

By Generic Flatness $\text{Spec } k \rightarrow \mathcal{X}$ is flat

$\Rightarrow$ its image $x \in |\mathcal{X}|$ is open. The open substack $\mathcal{G}_x$ is the

residual gerbe.

$\text{Spec } k \rightarrow \mathcal{G}_x$ is fppf $\Rightarrow \mathcal{G}_x$ is regular.

Assume $\mathcal{X}/k$ of f.t. $x \in \mathcal{X}(k)$ w/ affine sm. stab. G_x .

$$\exists \text{ monomorphism } BG_x^{\text{pre}} \rightarrow \mathcal{X}$$

$\left\{ \begin{array}{l} T \in \text{Sch}_k \hookrightarrow \text{a unique obj. of } BG_x^{\text{pre}} \text{ mapped to } T \rightarrow \text{Spec } k \xrightarrow{x} \mathcal{X} \\ \text{morphisms } T' \rightarrow T \mapsto T' \rightarrow BG_x^{\text{pre}} \text{ mapped to the corresponding morphisms in } \mathcal{X}. \end{array} \right.$

$$\text{Stackification } \rightsquigarrow BG_x \rightarrow \mathcal{X}$$

Repeat the argument for $\text{Spec } k \rightarrow \mathcal{X} \Rightarrow BG_x \hookrightarrow \mathcal{X}$ loc. closed.

$$\Rightarrow \mathcal{G}_x = BG_x. \quad \square$$

Rmk: The residual gerbe exists for any pt of a q -sep alg. stack.

Cor 3.5.20: $\mathcal{X}$ Noeth. alg. space $x \in |\mathcal{X}|$ f.t. pt. Then

x has a residue field $k(x)$ i.e.

$\exists$ a field $k(x)$ and a loc. closed immersion $\text{Spec } k(x) \hookrightarrow \mathcal{X}$ w/ image x .

Rmk: The residual gerbe exists for any pt of an alg. space.

Cor 3.5.18: $\mathcal{X}$ Noeth. alg. stack $x \in |\mathcal{X}|$ f.t. pt

$(U, u) \rightarrow (\mathcal{X}, x)$ sm. morphism from a scheme $U, u \in U$ f.t. pt

then $\exists$

$$\begin{array}{ccc} \mathcal{O}(u) & \hookrightarrow & U \\ \downarrow \square & & \downarrow \\ \mathcal{G}_x & \hookrightarrow & \mathcal{X} \end{array}$$

where $\mathcal{O}(u) \stackrel{\text{set}}{=} S(t^*(u))$, s.t. $R := U \times_{\mathcal{X}} U \rightrightarrows U$.

Eg: If $\mathcal{X} = [U/G]$ $u \in U(k)$ then

$$\mathcal{G}_u \hookrightarrow U$$

$$\downarrow \square \quad \downarrow$$

$$BG_x \hookrightarrow [U/G].$$